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XI. *Of the Fluents of Multinomials, and Series affected by radical Signs, which do not begin to converge till after the second Term; in a Letter from T. Simpson F. R. S. to W. Jones Esq; V. P. R. S.*

*Presented May 26.*  
1748.

**A**LTHO' the Application of infinite Series, and the Quadrature of the conic Sections, to the inverse Method of Fluxions has exercised the Pens of the most able Mathematicians, and produced many curious and useful Discoveries, yet nothing has been hitherto given, that I know of, whereby the Fluents of radical Multinomials and Series, which do not begin to converge till after the second Term, can be determined, so as to be of Use in the Solution of Problems: The common Method, by expanding the given Expression, being, you know, altogether impracticable in this Case.

The Consideration of which induced me to draw up the following Paper; which I humbly beg Leave to lay before you, who are so good a Judge of the various Improvements which this Subject has from time to time received.

What most encourages me to hope this little Essay will meet with your Approbation, is, that it is not merely an abstracted useless Speculation, but may be apply'd to good purpose in many difficult and important Enquiries into Nature; whereof I have put down one or two Instances, and shall further take the Liberty to observe here, that most of the  
lunar

Lunar Equations, given by Sir *Isaac Newton*, are only such Approximations as may be exhibited by the first Term of a Series derived by the Method here delivered.

*Proposition.*

*The Fluent of  $\overline{a + cz^n}^m \times z^{pn-1} \dot{z}$  being given (either in algebraic Terms, or from the Quadrature of the Conic Sections, &c.) it is proposed, by means thereof, to approximate the Fluent of  $\overline{a + cx^n + dx^{2n} + ex^{3n} + fx^{4n} \&c.}^m \times x^{pn-1} \dot{x}$ ; supposing the Series not to converge till after the second Term.*

Make  $cz^n = cx^n + dx^{2n} + ex^{3n} \&c.$  and let  $\mathcal{Q}$  be the given Fluent of  $\overline{a + cz^n}^m \times z^{pn-1} \dot{z}$ , answering to any proposed Value of  $x$ : Moreover let  $y = x^{pn}$ , or  $y^{\frac{1}{p}} = x^n$ , and let this Value of  $x^n$  be substituted in the first Equation, and it will become  $cz^n = cy^{\frac{1}{p}} + dy^{\frac{2}{p}} + ey^{\frac{3}{p}} \&c.$  whereof the Root  $y$  being extracted, we shall (by making  $R = -\frac{p d}{c}$ ,  $S = \frac{p \cdot p + 3}{2} \times \frac{d^2}{c^2} - \frac{p e}{c}$ ,  $T = \frac{-p \cdot p + 4 \cdot p + 5}{6} \times \frac{d^3}{c^3} + \frac{p \cdot p + 4}{1} \times \frac{d e}{c^2} - \frac{p f}{c} \&c.$ ) have  $y (x^{pn}) = z^{pn} + Rz^{pn+n} + Sz^{pn+2n} \&c.$  whence we also obtain  $x^{pn-1} \dot{x} = z^{pn-1} \dot{z} + \frac{p+1}{p} \times Rz^{pn+n-1} \dot{z} + \frac{p+2}{p} \times Sz^{pn+2n-1} \dot{z} \&c.$

Let this Value, with that of  $cx^n + dx^{2n} + ex^{3n} \&c.$  (above given) be now substituted in the proposed

posed Fluxion, and it will become  $\frac{a + cz^n}{a + cz^n} \times z^{pn-1} \dot{z} + \frac{p+1}{p} \times Rz^{pn+n-1} \dot{z} + \frac{p+2}{p} \times Sz^{pn+2n-1} \dot{z} \&c.$

Moreover, let  $v$  denote the Place, or Distance, of any Term, of this Expression, from the first (exclusive) then the Term itself (drawn into the common Multiplicator) will be denoted by  $\frac{a + cz^n}{a + cz^n} \times z^{p+v} \times Az^{pn+vn-1} \dot{z}$ ; and the Fluent thereof will

be truly expressed by  $\frac{p+1}{p+m+1} \times \frac{p+2}{p+m+2} \times \frac{p+3}{p+m+3} \times \dots$

$\frac{p+v}{p+m+v} \times \left[ \frac{a}{-c} \right]^v \times A Q + \frac{p+v \cdot A}{p} \times \frac{(a + cz^n)^{m+1} \times z^{pn-n}}{p+m+v \times nc}$  into

$z^{vn} - \frac{p+v-1}{p+m+v-1} \times \frac{az^{vn-1}}{c} + \frac{p+v-1 \cdot p+v-2}{p+v+m-1 \cdot p+v+m-2} \times \frac{a^2 z^{vn-2n}}{c^2} \&c.$  continued to as many Terms as there

are Units in  $v$ . Wherein let  $v$  be expounded by 1, 2, 3 &c. successively, and  $R, S, T, \&c.$  by  $A$  respectively: By which means the Fluent of the whole Expression will be obtained.

### Corol. I.

Because the Fluent of the general Term, when the Multiplicator  $\frac{a + cz^n}{a + cz^n} \times z^{m+1}$  becomes  $= 0$ , is barely

$= \frac{p+1}{p+m+1} \times \frac{p+2}{p+m+2} \times \frac{p+3}{p+m+3} \times \dots \times \frac{p+v}{p+m+v} \times \left[ \frac{a}{-c} \right]^v \times A Q$  the Fluent of the whole Expression will, therefore, in this Case be truly defined by

$$\mathcal{Q} \times 1 = \frac{p+1. Ra}{p+m+1. c} + \frac{p+1. p+2. Sa^2}{p+m+1. p+m+2. c^2} - \frac{p+1. p+2. p+3. Ta^3}{p+m+1. p+m+2. p+m+3. c^3} \&c.$$
 Where  $\mathcal{Q}$  denotes the Fluent of  $\overline{a+cz^n}^m \times z^{pn-1} \dot{z}$ , when  $z^n = \frac{a}{-c}$ .

*Corol. 2.*

But, if  $m+1$  and  $p$  be, each of them, the Half of an odd affirmative Number, and  $P$  be taken to denote the Periphery of a Circle whose Diameter is Unity, and  $-c$  be put  $=b$ , then the Value of  $\mathcal{Q}$  (or the Fluent of  $\overline{a-bz^n}^m \times z^{pn-1} \dot{z}$ , when  $z^n = \frac{a}{b}$ )

will be  $= a \frac{p+m}{nb^p} \times$

$\frac{1. 3. 5. 7 \&c. \text{ (to } p-\frac{1}{2} \text{ Factors)} \times 1. 3. 5. 7 \&c. \text{ to } (m+\frac{1}{2} \text{ Factors)}}{2. 4. 6. 8. 10. 12 \&c. \text{ (to } p+m \text{ Factors)}}$

Therefore the *Whole*, required, Fluent, of  $\overline{a-bx^n+dx^{2n}+ex^{3n}\&c.}^m \times x^{pn-1} \dot{x}$  is, in this Case, equal to the Product of that Expression into the following Series,  $1 + \frac{p+1. Ra}{p+m+1. b} + \frac{p+1. p+2. Sa^2}{p+m+1. p+m+2. b^2} \&c.$  Wherein  $R$  is to be taken  $= \frac{p.d}{b}$ ,  $S = \frac{p. p+3}{2} \times \frac{d^2}{b^2} + \frac{p.e}{b}$ ,  $T = \frac{p. p+1. p+5}{6} \times \frac{d^3}{b^3} + \frac{p. p+1}{1} \times \frac{d.e}{b^2} + \frac{p.f}{b}$ ,  $\&c.$  according to what is above specified.

The Use of what has been deliver'd above will, in some measure, appear from the Solution of the  
two

two following Problems, which I shall subjoin as Examples thereof. The first is;

*To find the Time of Oscillation in the Arch of a Cycloid, in a Medium resisting according to the duplicate Ratio of the Velocity.*

Let  $A$  denote the whole Arch of the Semi-Cycloid, or the Length of the Pendulum,  $a$  the Arch described in the whole Descent, and  $x$  any variable Part thereof described from the Beginning of the Descent; and let the Density of the Medium be, every-where, as  $\frac{1}{b}$ : Then the Fluxion of the Time will be found =

$$a - 1 + \frac{2a}{b} \times \frac{x}{2} - \frac{2x^2}{2.3b} + \frac{4x^3}{2.3.4b^2} - \frac{8x^4}{2.3.4.5b^3} \& c. \Bigg|^{-\frac{1}{2}} \times$$

$$\frac{1}{2A} \Big|^{-\frac{1}{2}} \times x^{\frac{1}{2}} \dot{x} * : \text{ which being compared with } \frac{a - bx^n + dx^{2n} + ex^{3n} \& c. \Big|^m \times x^{pn-1} \dot{x} \text{ (vide Corol. 2.)}$$

we shall, in this Case, have  $n=1$ ,  $m=-\frac{1}{2}$ ,  $p=\frac{1}{2}$ ,

$$a=a, b=1 \times \frac{2a}{b} \times \frac{1}{2}, \frac{d}{b}=\frac{2}{3b}, \frac{e}{b}=-\frac{1}{3b^2}, \frac{f}{b}=\frac{2}{15b^3} \& c.$$

Whence  $R=\frac{1}{3b}$ ,  $S=\frac{2}{9b^2}$ ,  $T=\frac{8}{45b^3} \& c.$  Also

$$\frac{a^{p+m} P}{nb^p} \times \frac{1.3.5.7(p-\frac{1}{2}) \times 1.3.5.7(m+\frac{1}{2})}{2.4.6.8.10.(p+m)} = \frac{P}{b^{\frac{1}{2}}}, \text{ and}$$

$$1 + \frac{p+1.Ra}{p+m+1.b} + \frac{p+1.p+2.Sa^2}{p+m+1.p+m+2.b^2} \& c. = 1 + \frac{a}{2bb} + \frac{5a^2}{12b^2b^2}$$

\* The Investigation of this, and the Fluxion in the following Example, are both given in my Essays.



that Distance be denoted by Unity; and, supposing  $z$  to denote any other Distance, let the centripetal Force be universally expressed by  $z^n$ . Then the Fluxion of the Angle at the Center will be expressed by

$$\frac{-pz}{x\sqrt{p^2 + \frac{2}{n+1}xz - p^2 - \frac{2z}{n+1}^{n+3}}} \quad \text{Put } a = 1 - p^2, \quad v = \frac{n+3}{2} \text{ and } x = 1 - z^2, \text{ and it will become}$$

$$\frac{\frac{1}{2}\sqrt{1-a} \times \dot{x}}{1-x \times \sqrt{ax + \frac{1-vx-1-x^v}{1-v}}} = \frac{1}{2} 1 - a^{\frac{1}{2}} \text{ into}$$

$$a - \frac{vx}{2} + \frac{v \cdot v-2}{2 \cdot 3} x x^2 - \frac{v \cdot v-2 \cdot v-3}{2 \cdot 3 \cdot 4} x x^3 \&c. \Big|^{-\frac{1}{2}} x$$

$$x^{-\frac{1}{2}} \dot{x} + x^{\frac{1}{2}} \dot{x} + x^{\frac{3}{2}} \dot{x} \&c.$$

Now, to find the Fluent of the first Term hereof (drawn into the general Multiplicator) or

$$a - \frac{vx}{2} + \frac{v \cdot v-2}{2 \cdot 3} x x^2 \&c. \Big|^{-\frac{1}{2}} x x^{-\frac{1}{2}} \dot{x}, \text{ we have (as}$$

$$\text{before) } n=1, m=-\frac{1}{2}, p=\frac{1}{2}, b=\frac{v}{2}, \frac{d}{b}=\frac{v-2}{3},$$

$$\frac{e}{b}=-\frac{v-2 \cdot v-3}{3 \cdot 4}, \&c. \text{ Also } R=\frac{v-2}{6}, S=\frac{v-2 \cdot 4v-5}{72};$$

and consequently the Fluent itself (when the Body

$$\text{arrives at the lower Apse) } = \frac{P}{\sqrt{\frac{1}{2}v}} x$$

$$+ \frac{v-2}{2v} xa + \frac{5 \cdot v-2 \cdot 4v-5}{48v^2} xa^2 + \frac{7 \cdot v-v \cdot 16v^2-37v+22}{6 \cdot 48v^3}$$

&c. After the same manner the Fluent of the second



the second Term will come out =  $\frac{P}{\sqrt{\frac{1}{2}v}} \times$

$$\frac{a}{v} + \frac{5 \cdot v-2}{4v^2} \times a^2 + \frac{3 \cdot v-2 \cdot v-3}{48v^3} \times a^3 \&c. \text{ that of the}$$

$$\text{third} = \frac{P}{\sqrt{\frac{1}{2}v}} \times \frac{3a^2}{2v^2} + \frac{35 \cdot v-2}{12v^3} \times a^3 \&c. \&c. \&c.$$

Whence, by collecting these several Fluents together, we have  $\frac{P}{\sqrt{\frac{1}{2}v}} \times$

$$1 + \frac{1}{2}a + \frac{20v^2-5v+2}{48v^2} \times a^2 + \frac{112v^3-53v^2+42v-8}{6 \cdot 48v^3} \times a^3 \&c.$$

for the Fluent of the whole Expression: And this,

drawn into  $\frac{1}{2} \times -1 - \frac{a}{2} - \frac{a^2}{8} \&c. (= \frac{1}{2} \times 1 - a^{\frac{1}{2}})$  will be

$$= \frac{P}{\sqrt{2v}} \times 1 * \frac{v-2 \cdot 2v-1}{48} \times \frac{a^2}{v^2} + \frac{v-2 \cdot 2v-1}{72} \times \frac{a^3}{v^3} \&c.$$

$$= \frac{P}{\sqrt{n+3}} \times 1 * \frac{n-1 \cdot n+2}{24} \times \frac{a^2}{n+3} + \frac{n-1 \cdot n+2}{18} \times \frac{a^3}{n+3} \&c.$$

which, in Degrees, gives  $\frac{180}{\sqrt{n+3}} \times$

$$1 + \frac{n-1 \cdot n+2}{24} \times \frac{a^2}{n+3} + \frac{n-1 \cdot n+2}{18} \times \frac{a^3}{n+3} \&c. \text{ for}$$

the true Measure of the Angle required.